Singapore Examinations and Assessment Board

Cambridge Assessment International Education

Singapore–Cambridge General Certificate of Education Advanced Level Higher 3 (2023)

Mathematics (Syllabus 9820)

CONTENTS

PREAMBLE

Mathematicians work with precise definitions, make conjectures, prove new results and solve problems. They are concerned with the properties of mathematical objects and the applications of abstract ideas and models to solve problems. Mathematical truths and solutions come from rigorously constructed arguments called proofs and mathematically sound procedures and steps. The work of mathematicians has impact in different fields, far beyond just sciences and engineering.

H3 Mathematics provides students, who intend to pursue mathematics at the university, with an insight into the practice of a mathematician. It equips students with the knowledge and skills to understand and write mathematical statements, proofs and solutions, and the techniques and results that come in helpful in their work. Students will develop these competencies through proving mathematical results and solving non-routine mathematical problems in the course of the learning.

SYLLABUS AIMS

The aims of H3 Mathematics are to enable students to:

(a) acquire advanced problem-solving skills and methods of proof by learning useful mathematical results and techniques to solve non-routine problems and prove statements

(b) develop rigour in mathematical argument and precision in the use of mathematical language through the writing and evaluation of mathematical proofs and solutions

(c) experience and appreciate the practice, value and rigour of mathematics as a discipline.

ASSESSMENT OBJECTIVES (AO)

There are three levels of assessment objectives for the examination.

The assessment will test candidates' abilities to:

AO1 Understand and apply mathematical concepts, skills and results to solve non-routine problems, including those that may require integration of concepts and skills from more than one topic.

AO2 Understand and apply advanced methods and techniques of proof to establish the truth or falsity of a mathematical statement.

AO3 Reason and communicate in precise mathematical language through the writing and evaluation of mathematical proofs and solutions.

USE OF A GRAPHING CALCULATOR (GC)

The use of an approved GC without computer algebra system will be expected. The examination papers will be set with the assumption that candidates will have access to GC. As a general rule, unsupported answers obtained from GC are allowed unless the question states otherwise. Where unsupported answers from GC are not allowed, candidates are required to present the mathematical steps using mathematical notations and not calculator commands. For questions where graphs are used to find a solution, candidates should sketch these graphs as part of their answers. Incorrect answers without working will receive no marks. However, if there is written evidence of using GC correctly, method marks may be awarded.

Students should be aware that there are limitations inherent in GC. For example, answers obtained by tracing along a graph to find roots of an equation may not produce the required accuracy.

LIST OF FORMULAE AND STATISTICAL TABLES

Candidates will be provided in the examination with a list of formulae and statistical tables.

INTEGRATION AND APPLICATION

Notwithstanding the presentation of the topics in the syllabus document, it is envisaged that some examination questions may integrate ideas from more than one topic, and that topics may be tested in the contexts of problem solving and application of mathematics.

SCHEME OF EXAMINATION PAPERS

For the examination in H3 Mathematics, there will be one 3-hour paper marked out of 100. The paper will consist of 8 to 10 questions of different lengths, and each question is worth 9 to 16 marks.

Candidates will be expected to answer **all** questions.

CONTENT OUTLINE

Knowledge of the content of H2 Mathematics is assumed.

Students will learn to prove properties and results, and solve non-routine problems involving:

(1) H2 Mathematics content areas

- **(a) Functions**, e.g. graphs, symmetries, derivatives, integrals, differential equations, limiting behaviours, bounds.

- **(b) Sequences and series**, e.g. general terms, sum, limiting behaviours, bounds.

The examples in **(a)** and **(b)** above illustrate some types of problems that are based on content in H2 Mathematics.

(2) Additional content areas

(a) Inequalities: AM–GM inequality, Cauchy-Schwarz inequality, triangle inequality.

(b) Numbers: primes, coprimes, divisibility, greatest common divisor, division algorithm, congruences and modular arithmetic.

(c) Counting: distribution problems, Stirling numbers of the second kind, recurrence equations, bijection principle, principle of inclusion and exclusion.

The above define the expected scope of content knowledge that may be assessed.

Notwithstanding the content areas defined above, students will also prove results and solve problems outside these defined areas or at the intersection of two or more such areas using their ability to understand and apply given definitions or results.

Mathematical Skills

Students are expected to apply the following skills:

Skills	**Examples**
a) Communicate mathematical ideas using mathematical language	• Terms such as ‘Definition’ and ‘Theorem’ • Conditional statements (such as ‘if *P* then *Q*’ and ‘*P* if and only if *Q*’) • Necessary and sufficient conditions • Existential and universal quantifiers (such as ‘there exists’, ‘for each’, ‘for all’) • Logical connectives (such as ‘and’, ‘or’, ‘not’, ‘implies’) • Converse, inverse, contrapositive and negation of statements • Set notation and language
b) Develop and critically evaluate mathematical arguments using mathematical reasoning principles, including methods of proof	• Direct proof • Proof by mathematical induction • Disproof by counterexample • Proof by contradiction • Proof of existence • Proof by construction • Pigeonhole principle • Symmetry principle
c) Solve mathematical problems using problem solving heuristics	• Working backwards • Uncovering pattern and structure • Solving a simpler/similar problem • Considering cases • Restating the problem (e.g. contrapositive)

Mathematical Results

Students may use the following theorems and results without proof. In addition, they may be required to use the ideas in the proofs of these theorems and results to solve other problems.

(i) (AM-GM inequality) For any nonnegative real numbers $x_1, x_2, \ldots, x_n$,

$$\frac{x_1 + x_2 + \ldots + x_n}{n} \geqslant \sqrt[n]{x_1 x_2 \ldots x_n},$$

where the equality holds if and only if $x_1 = x_2 = \ldots = x_n$.

(ii) (Cauchy-Schwarz inequality) For any real numbers $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$,

$$\left(\sum_{i=1}^{n} u_i v_i\right)^2 \leqslant \left(\sum_{i=1}^{n} u_i^2\right)\left(\sum_{i=1}^{n} v_i^2\right),$$

where the equality holds if there exists a nonzero constant k such that $u_i = kv_i$ for all $i = 1, 2, \ldots, n$.

(iii) (Triangle inequality) For any real numbers $x_1, x_2, \ldots, x_n$,

$$\left|x_1 + x_2 + \ldots + x_n\right| \leqslant \left|x_1\right| + \left|x_2\right| + \ldots + \left|x_n\right|,$$

where the equality holds if $x_1, x_2, \ldots, x_n$ are all nonnegative.

(iv) (The Fundamental Theorem of Arithmetic) Every integer $n > 1$ can be expressed as a product of primes in a unique way apart from the order of the prime factors.

(v) There exist infinitely many primes.

(vi) (Division Algorithm) Let a be an integer and b a positive integer. Then there exists unique integers q and r, with $0 \leqslant r < b$, such that $a = bq + r$.

(vii) If a and b are positive integers, then their greatest common divisor (gcd) is a linear combination of a and b, that is there exist integers s and t such that $\gcd(a, b) = sa + tb$.

(viii) If a and b are positive integers, and there exist integers s and t such that $sa + tb = 1$, then a and b are coprime.

Knowledge of the following theorems is not required and should not be used without proof: Euclidean algorithm, Chinese remainder theorem, Wilson's Theorem, Fermat's little theorem, and Euler's theorem.

MATHEMATICAL NOTATION

The list which follows summarises the notation used in Cambridge's Mathematics examinations. Although primarily directed towards A-Level, the list also applies, where relevant, to examinations at all other levels.

1. *Set Notation*

$\in$	is an element of
$\notin$	is not an element of
$\{x_1, x_2, \ldots\}$	the set with elements $x_1, x_2, \ldots$
$\{x: \ldots\}$	the set of all x such that
$\mathrm{n}(A)$	the number of elements in set A
$\varnothing$	the empty set
$\mathscr{E}$	universal set
A'	the complement of the set A
$\mathbb{Z}$	the set of integers, $\{0, \pm1, \pm2, \pm3, \ldots\}$
$\mathbb{Z}^+$	the set of positive integers, $\{1, 2, 3, \ldots\}$
$\mathbb{Q}$	the set of rational numbers
$\mathbb{Q}^+$	the set of positive rational numbers, $\{x \in \mathbb{Q}: x > 0\}$
$\mathbb{Q}_0^+$	the set of positive rational numbers and zero, $\{x \in \mathbb{Q}: x \geqslant 0\}$
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^+$	the set of positive real numbers, $\{x \in \mathbb{R}: x > 0\}$
$\mathbb{R}_0^+$	the set of positive real numbers and zero, $\{x \in \mathbb{R}: x \geqslant 0\}$
$\mathbb{R}^n$	the real n-tuples
$\mathbb{C}$	the set of complex numbers
$\subseteq$	is a subset of
$\subset$	is a proper subset of
$\nsubseteq$	is not a subset of
$\not\subset$	is not a proper subset of
$\cup$	union
$\cap$	intersection
$[a, b]$	the closed interval $\{x \in \mathbb{R}: a \leqslant x \leqslant b\}$
$[a, b)$	the interval $\{x \in \mathbb{R}: a \leqslant x < b\}$
$(a, b]$	the interval $\{x \in \mathbb{R}: a < x \leqslant b\}$
(a, b)	the open interval $\{x \in \mathbb{R}: a < x < b\}$

2. *Miscellaneous Symbols*

$=$	is equal to
$\neq$	is not equal to
$\equiv$	is identical to or is congruent to
$\approx$	is approximately equal to
$\propto$	is proportional to
$<$	is less than
$\leqslant$; $\ngtr$	is less than or equal to; is not greater than
$>$	is greater than
$\geqslant$; $\nless$	is greater than or equal to; is not less than
∞	infinity

3. *Operations*

$a + b$	a plus b
$a - b$	a minus b
$a \times b$, ab, $a.b$	a multiplied by b
$a \div b$, $\frac{a}{b}$, a/b	a divided by b
$a : b$	the ratio of a to b
$\sum_{i=1}^{n} a_i$	$a_1 + a_2 + ... + a_n$
$\sqrt{a}$	the positive square root of the real number a
$\lvert a \rvert$	the modulus of the real number a
$n!$	n factorial for $n \in \mathbb{Z}^+ \cup \{0\}$, $(0! = 1)$
$\binom{n}{r}$	the binomial coefficient $\frac{n!}{r!(n-r)!}$, for $n, r \in \mathbb{Z}^+ \cup \{0\}$, $0 \leqslant r \leqslant n$ $\frac{n(n-1)...(n-r+1)}{r!}$, for $n \in \mathbb{Q}$, $r \in \mathbb{Z}^+ \cup \{0\}$

4. *Functions*

f	the function f
$f(x)$	the value of the function f at x
$f: A \to B$	f is a function under which each element of set A has an image in set B
$f: x \mapsto y$	the function f maps the element x to the element y
f^{-1}	the inverse of the function f
$g \circ f$, gf	the composite function of f and g which is defined by $(g \circ f)(x)$ or $gf(x) = g(f(x))$
$\lim_{x \to a} f(x)$	the limit of $f(x)$ as x tends to a
Δx ; δx	an increment of x
$\frac{dy}{dx}$	the derivative of y with respect to x
$\frac{d^n y}{dx^n}$	the nth derivative of y with respect to x
$f'(x), f''(x), \ldots, f^{(n)}(x)$	the first, second, … nth derivatives of $f(x)$ with respect to x
$\int y \, dx$	indefinite integral of y with respect to x
$\int_a^b y \, dx$	the definite integral of y with respect to x for values of x between a and b
$\dot{x}, \ddot{x}, \ldots$	the first, second, …derivatives of x with respect to time

5. *Exponential and Logarithmic Functions*

e	base of natural logarithms
e^x, exp x	exponential function of x
$\log_a x$	logarithm to the base a of x
$\ln x$	natural logarithm of x
$\lg x$	logarithm of x to base 10

6. *Circular Functions and Relations*

sin, cos, tan, cosec, sec, cot	the circular functions
$\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$ cosec^{-1}, $\sec^{-1}$, $\cot^{-1}$	the inverse circular functions

7. *Complex Numbers*

i	the square root of –1
z	a complex number, $z = x + \mathrm{i}y$
	$= r(\cos\theta + \mathrm{i}\sin\theta),\ r \in \mathbb{R}_0^+$
	$= r\mathrm{e}^{\mathrm{i}\theta},\ r \in \mathbb{R}_0^+$
Re z	the real part of z, $\mathrm{Re}\,(x + \mathrm{i}y) = x$
Im z	the imaginary part of z, $\mathrm{Im}\,(x + \mathrm{i}y) = y$
$\lvert z \rvert$	the modulus of z, $\lvert x + \mathrm{i}y \rvert = \sqrt{x^2 + y^2}$, $\lvert r(\cos\theta + \mathrm{i}\sin\theta) \rvert = r$
arg z	the argument of z, $\arg(r(\cos\theta + \mathrm{i}\sin\theta)) = \theta$, $-\pi < \theta \leqslant \pi$
z^*	the complex conjugate of z, $(x + \mathrm{i}y)^* = x - \mathrm{i}y$

8. *Matrices*

$\mathbf{M}$	a matrix $\mathbf{M}$
$\mathbf{M}^{-1}$	the inverse of the square matrix $\mathbf{M}$
$\mathbf{M}^{\mathrm{T}}$	the transpose of the matrix $\mathbf{M}$
det $\mathbf{M}$	the determinant of the square matrix $\mathbf{M}$

9. *Vectors*

$\mathbf{a}$	the vector $\mathbf{a}$
$\overrightarrow{AB}$	the vector represented in magnitude and direction by the directed line segment AB
$\hat{\mathbf{a}}$	a unit vector in the direction of the vector $\mathbf{a}$
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	unit vectors in the directions of the cartesian coordinate axes
$\lvert \mathbf{a} \rvert$	the magnitude of $\mathbf{a}$
$\lvert \overrightarrow{AB} \rvert$	the magnitude of $\overrightarrow{AB}$
$\mathbf{a.b}$	the scalar product of $\mathbf{a}$ and $\mathbf{b}$
$\mathbf{a} \times \mathbf{b}$	the vector product of $\mathbf{a}$ and $\mathbf{b}$

10. *Probability and Statistics*

A, B, C, etc.	events
$A \cup B$	union of events A and B
$A \cap B$	intersection of the events A and B
$\mathrm{P}(A)$	probability of the event A
A'	complement of the event A, the event 'not A'
$\mathrm{P}(A \mid B)$	probability of the event A given the event B
X, Y, R, etc.	random variables
x, y, r, etc.	value of the random variables X, Y, R, etc.
$x_1, x_2, \ldots$	observations
$f_1, f_2, \ldots$	frequencies with which the observations, $x_1, x_2, \ldots$occur
$\mathrm{p}(x)$	the value of the probability function $\mathrm{P}(X = x)$ of the discrete random variable X
$p_1, p_2, \ldots$	probabilities of the values $x_1, x_2, \ldots$of the discrete random variable X
$\mathrm{f}(x), \mathrm{g}(x)\ldots$	the value of the probability density function of the continuous random variable X
$\mathrm{F}(x), \mathrm{G}(x)\ldots$	the value of the (cumulative) distribution function $\mathrm{P}(X \leqslant x)$ of the random variable X
$\mathrm{E}(X)$	expectation of the random variable X
$\mathrm{E}[\mathrm{g}(X)]$	expectation of $\mathrm{g}(X)$
$\mathrm{Var}(X)$	variance of the random variable X
$\mathrm{B}(n, p)$	binomial distribution, parameters n and p
$\mathrm{Po}(\mu)$	Poisson distribution, mean μ
$\mathrm{N}(\mu, \sigma^2)$	normal distribution, mean μ and variance σ^2
μ	population mean
σ^2	population variance
σ	population standard deviation
$\overline{x}$	sample mean
s^2	unbiased estimate of population variance from a sample, $s^2 = \dfrac{1}{n-1}\Sigma(x - \overline{x})^2$
φ	probability density function of the standardised normal variable with distribution $\mathrm{N}(0, 1)$
Φ	corresponding cumulative distribution function
ρ	linear product-moment correlation coefficient for a population
r	linear product-moment correlation coefficient for a sample