Further Mathematics

Singapore-Cambridge General Certificate of Education Advanced Level Higher 2

(Syllabus 9649)

(Updated for examination from 2021)

CONTENTS

Significant changes to the syllabus are indicated by black vertical lines either side of the text.

Singapore Examinations and Assessment Board

Cambridge Assessment International Education

PREAMBLE

Mathematics drives many of the advancements in sciences, engineering, economics and technology. It is at the heart of many of the innovative products and services today. A strong grounding in mathematics is essential for students who aspire to be scientists or engineers or any other professionals who require mathematical tools to solve complex problems.

H2 Further Mathematics is designed for students who are mathematically-inclined and who intend to specialise in mathematics, sciences or engineering or disciplines with higher demand on mathematical skills. It extends and expands on the range of mathematics and statistics topics in H2 Mathematics and provides these students with a head start in learning a wider range of mathematical methods and tools that are useful for solving more complex problems in mathematics and statistics.

H2 Further Mathematics is to be offered with H2 Mathematics as a double mathematics course.

SYLLABUS AIMS

The aims of H2 Further Mathematics are to enable students to:

(a) acquire a wider range of mathematical concepts and stronger set of mathematical skills for their tertiary studies in mathematics, sciences, engineering and other related disciplines with a heavier demand on mathematics

(b) develop thinking, reasoning, communication and modelling skills through a mathematical approach to problem-solving

(c) connect ideas within mathematics and apply mathematics in the contexts of sciences, engineering and other related disciplines

(d) experience and appreciate the rigour and abstraction in the discipline.

ASSESSMENT OBJECTIVES (AO)

There are three levels of assessment objectives for the examination.

The assessment will test candidates' abilities to:

AO1 Understand and apply a wide range of mathematical concepts and skills in a variety of problems, including those that may be set in unfamiliar contexts, or require integration of concepts and skills from more than one topic.

AO2 Formulate real-world problems mathematically, solve the mathematical problems, interpret and evaluate the mathematical solutions in the context of the problems.

AO3 Reason and communicate mathematically through forming conjectures, making deductions and constructing rigorous mathematical arguments and proofs.

USE OF A GRAPHING CALCULATOR (GC)

The use of an approved GC without computer algebra system will be expected. The examination papers will be set with the assumption that candidates will have access to GC. As a general rule, unsupported answers obtained from GC are allowed unless the question states otherwise. Where unsupported answers from GC are not allowed, candidates are required to present the mathematical steps using mathematical notations and not calculator commands. For questions where graphs are used to find a solution, candidates should sketch these graphs as part of their answers. Incorrect answers without working will receive no marks. However, if there is written evidence of using GC correctly, method marks may be awarded.

Students should be aware that there are limitations inherent in GC. For example, answers obtained by tracing along a graph to find roots of an equation may not produce the required accuracy.

LIST OF FORMULAE AND STATISTICAL TABLES

Candidates will be provided in the examination with a list of formulae and statistical tables.

INTEGRATION AND APPLICATION

Notwithstanding the presentation of the topics in the syllabus document, it is envisaged that some examination questions may integrate ideas from more than one topic, and that topics may be tested in the contexts of problem solving and application of mathematics.

Possible list of H2 Further Mathematics applications and contexts:

Applications and contexts	**Some possible topics involved**
Kinematics and dynamics (e.g. free fall, projectile motion, orbital motion, collisions)	Functions; Calculus; Vectors
Movie graphics	Vectors
Optics (design of mirrors)	Functions; Conic Sections
Optimisation problems (e.g. maximising strength, minimising surface area)	Inequalities; System of linear equations; Calculus
Electrical circuits (including alternating current circuit)	Complex numbers; Calculus
Population growth (e.g. spread of diseases), radioactive decay, heating and cooling problems, mixing, chemical changes, charging	Differential equations
Search engines, cryptography, digital music	Matrices and linear spaces
Financial maths (e.g. banking, insurance)	Sequences and series; Probability; Sampling distributions
Standardised testing	Normal distribution; Probability
Market research (e.g. consumer preferences, product claims)	Sampling distributions; Hypothesis testing; Correlation and regression

Applications and contexts	Some possible topics involved
Clinical research (e.g. correlation studies)	Sampling distributions; Hypothesis testing; Correlation and regression
Polling	Confidence intervals; Hypothesis testing
Genetics	Chi-square tests

The list illustrates some types of contexts in which the mathematics learnt in the syllabus may be applied, and is by no means exhaustive. While problems may be set based on these contexts, no assumptions will be made about the knowledge of these contexts. All information will be self-contained within the problem.

SCHEME OF EXAMINATION PAPERS

For the examination in H2 Further Mathematics, there will be two 3-hour papers, each carrying 50% of the total mark, and each marked out of 100, as follows:

PAPER 1 (3 hours)
A paper consisting of 10 to 12 questions of different lengths and marks based on the Pure Mathematics section of the syllabus.

There will be at least two questions on application of Mathematics in real-world contexts, including those from sciences and engineering. Each question will carry at least 12 marks and may require concepts and skills from more than one topic.

Candidates will be expected to answer **all** questions.

PAPER 2 (3 hours)
A paper consisting of two sections, Sections A and B.

Section A (Pure Mathematics – 50 marks) will consist of 5 to 6 questions of different lengths and marks based on the Pure Mathematics section (i.e. Algebra and Calculus, and Discrete Mathematics, Matrices and Numerical Methods) of the syllabus.

Section B (Probability and Statistics – 50 marks) will consist of 5 to 6 questions of different lengths and marks based on the Probability and Statistics section of the syllabus.

There will be at least two questions in Section B on application of Mathematics in real-world contexts, including those from sciences and engineering. Each question will carry at least 12 marks and may require concepts and skills from more than one topic.

Candidates will be expected to answer **all** questions.

CONTENT OUTLINE

Knowledge of the content of H2 Mathematics is assumed.

	Topic/Sub-topics	Content
SECTION A: PURE MATHEMATICS		
1	**Algebra and Calculus**	
1.1	Mathematical induction	Include: • use of method of mathematical induction to establish a given result involving series and recurrence relations, derivatives, inequalities, or divisibility • formulation of conjectures
1.2	Complex numbers	Include: • geometrical effects of conjugation, addition, subtraction, multiplication and division of complex numbers • loci of simple equations and inequalities such as $\|z-c\|\leqslant r$, $\|z-a\| = \|z-b\|$ and $\arg(z-a) = \alpha$ (excluding loci of $\|z-a\| = k\|z-b\|$, where $k \neq 1$ and $\arg(z-a) - \arg(z-b) = \alpha$) • use of de Moivre's theorem to find the powers and nth roots of a complex number, and to derive trigonometric identities
1.3	Polar coordinates	Include: • simple polar curves (for $0 \leqslant \theta < 2\pi$ or $-\pi < \theta \leqslant \pi$ or a subset of either of these intervals; and where r is non-negative throughout the domain) • use of formula $A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \,\mathrm{d}\theta$ for the area of a sector • arc length of curves defined in polar form
1.4	Conic sections	Include: • equation of conic sections in the cartesian form $Ax^2 + By^2 + Cx + Dy + E = 0$ • focus, directrix and eccentricity • geometrical properties of conic sections: – sum of the distance from a point on an ellipse to a pair of fixed points is a constant – difference between distances from a point on a hyperbola to a pair of fixed points is a constant – reflective properties of conic sections • conic sections in polar form given by $r = \frac{ep}{1 \pm e\cos\theta}$ or $r = \frac{ep}{1 \pm e\sin\theta}$, where $e > 0$ is the eccentricity and $\|p\|$ is the distance between the focus (pole) and the directrix

	Topic/Sub-topics	Content
1.5	Applications of definite integrals	Include: • arc length of curves defined in cartesian or parametric form • volume of revolution about the *x*- or *y*-axis for curves defined in cartesian or parametric form using discs or shells as appropriate • surface area of revolution about the *x*- or *y*-axis for curves defined in cartesian or parametric form
1.6	Differential equations	Include: • analytical solution of first order and second order linear differential equations of the form: (i) $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x)\,\mathrm{g}(y)$ (ii) $\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{p}(x)y = \mathrm{q}(x)$, using an integrating factor (iii) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + a\frac{\mathrm{d}y}{\mathrm{d}x} + by = 0$ (iv) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + a\frac{\mathrm{d}y}{\mathrm{d}x} + by = \mathrm{f}(x)$, where $\mathrm{f}(x)$ is a polynomial or $p\mathrm{e}^{kx}$ or $p\cos(kx) + q\sin(kx)$ including those that can be reduced to the above by means of a given substitution • relationship between the solution of a non-homogenous equation and the associated homogenous equation • family of solution curves • exponential growth model • logistic growth model, equilibrium points and their stability, and harvesting Exclude phase lines, slope fields and bifurcation diagrams.
2	**Discrete Mathematics, Matrices and Numerical Methods**	
2.1	Recurrence relations	Include: • sequence generated by a simple recurrence relation including the use of a graphing calculator to generate the sequence defined by the recurrence relation • behaviour of a sequence, such as the limiting behaviour of a sequence • solution of (i) first order linear (homogeneous and non-homogeneous) recurrence relations with constant coefficients of the form $u_n = au_{n-1} + b$, $a, b \in \mathbb{R}$, $a \neq 0$ (ii) second order linear homogeneous recurrence relations with constant coefficients • modelling with recurrence relations of the forms above

	Topic/Sub-topics	Content
2.2	Matrices and linear spaces	Include: • use of matrices to represent a set of linear equations • operations on 3×3 matrices • determinant of a square matrix and inverse of a non-singular matrix (2×2 and 3×3 matrices only) • use of matrices to solve a set of linear equations (including row reduction and echelon forms, and geometrical interpretation of the solution) • linear spaces and subspaces, and the axioms (restricted to spaces of finite dimension over the field of real numbers only) • linear independence and span • basis and dimension (in simple cases), including use of terms such as 'column space', 'row space', 'range space' and 'null space' • rank of a square matrix and relation between rank, dimension of null space and order of the matrix • linear transformations and matrices from $\mathbb{R}^n \to \mathbb{R}^m$ • eigenvalues and eigenvectors of square matrices (2×2 and 3×3 matrices, restricted to cases where the eigenvalues are real and distinct) • diagonalisation of a square matrix M by expressing the matrix in the form $\mathbf{QDQ}^{-1}$, where $\mathbf{D}$ is a diagonal matrix of eigenvalues and $\mathbf{Q}$ is a matrix whose columns are eigenvectors, and use of this expression such as to find the powers of M
2.3	Numerical methods	Include: • location of roots of an equation by simple graphical or numerical methods • approximation of roots of equations using linear interpolation and Newton-Raphson method including cases where each method fails to converge to the required root • iterations involving recurrence relations of the form $x_{n+1} = \mathrm{F}(x_n)$ including cases where the method fails to converge • approximation of integral of a function using the trapezium rule and Simpson's rule • approximation of solutions of first order differential equations using the Euler method (including the use of the improved Euler formula)

	Topic/Sub-topics	Content
SECTION B: PROBABILITY AND STATISTICS		
3	**Probability and Statistics**	
3.1	Discrete random variables	Include: • use of Poisson distribution Po(μ) and geometric distribution Geo(p) as probability models, including conditions under which each distribution is a suitable model • mean and variance for Poisson and geometric distributions • additive property of the Poisson distribution
3.2	Continuous random variables	Include: • probability density function of a continuous random variable and its mean and variance (includes 'piecewise' probability density function) • cumulative distribution function and its relationship with the probability density function • concepts of median and mode of a continuous random variable • use of the result $\mathrm{E}(\mathrm{g}(X)) = \int_{-\infty}^{\infty} \mathrm{g}(x)\,\mathrm{f}(x)\,\mathrm{d}x$ in simple cases, where f(x) is the probability density function of X and g(x) is a function of X • uniform distribution and exponential distribution as probability models • relationship between Poisson and exponential distributions

	Topic/Sub-topics	Content
3.3	Hypothesis testing and Confidence intervals	Include: • formulation of hypotheses and testing for a population mean using a small sample drawn from a normal population of unknown variance using a *t*-test • formulation of hypotheses for the difference of population means, and apply, as appropriate: – a 2-sample *t*-test – a paired sample *t*-test – a test using a normal distribution • contingency tables and χ^2 -tests of: – goodness of fit – independence (excluding Yates' correction for continuity) • connection between confidence interval and hypothesis test • confidence interval for the population mean based on: – a random sample from a normal population of known variance – a small random sample drawn from a normal population of unknown variance – a large random sample from any population • confidence interval for population proportion (including concept of sample proportion) from a large random sample • interpretation of confidence intervals and the results of a hypothesis test in the context of the problem Exclude the use of the term 'Type I error', concept of Type II error and power of a test.
3.4	Non-parametric tests	Include: • formulation of hypotheses and testing for: – a population median using Sign test • identical probability distributions for two sampled populations in a paired difference design using Wilcoxon matched-pair signed rank test • advantages and disadvantages of non-parametric tests Exclude treatment of tied ranks.

MATHEMATICAL NOTATION

The list which follows summarises the notation used in Cambridge's Mathematics examinations. Although primarily directed towards A-Level, the list also applies, where relevant, to examinations at all other levels.

1. *Set Notation*

$\in$	is an element of
$\notin$	is not an element of
$\{x_1, x_2, \ldots\}$	the set with elements $x_1, x_2, \ldots$
$\{x: \ldots\}$	the set of all x such that
$\mathrm{n}(A)$	the number of elements in set A
$\varnothing$	the empty set
$\mathscr{E}$	universal set
A'	the complement of the set A
$\mathbb{Z}$	the set of integers, $\{0, \pm1, \pm2, \pm3, \ldots\}$
$\mathbb{Z}^+$	the set of positive integers, $\{1, 2, 3, \ldots\}$
$\mathbb{Q}$	the set of rational numbers
$\mathbb{Q}^+$	the set of positive rational numbers, $\{x \in \mathbb{Q}: x > 0\}$
$\mathbb{Q}_0^+$	the set of positive rational numbers and zero, $\{x \in \mathbb{Q}: x \geqslant 0\}$
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^+$	the set of positive real numbers, $\{x \in \mathbb{R}: x > 0\}$
$\mathbb{R}_0^+$	the set of positive real numbers and zero, $\{x \in \mathbb{R}: x \geqslant 0\}$
$\mathbb{R}^n$	the real n-tuples
$\mathbb{C}$	the set of complex numbers
$\subseteq$	is a subset of
$\subset$	is a proper subset of
$\nsubseteq$	is not a subset of
$\not\subset$	is not a proper subset of
$\cup$	union
$\cap$	intersection

$[a, b]$	the closed interval $\{x \in \mathbb{R}: a \leqslant x \leqslant b\}$
$[a, b)$	the interval $\{x \in \mathbb{R}: a \leqslant x < b\}$
$(a, b]$	the interval $\{x \in \mathbb{R}: a < x \leqslant b\}$
(a, b)	the open interval $\{x \in \mathbb{R}: a < x < b\}$

2. *Miscellaneous Symbols*

$=$	is equal to
$\neq$	is not equal to
$\equiv$	is identical to or is congruent to
$\approx$	is approximately equal to
$\propto$	is proportional to
$<$	is less than
$\leqslant$; $\ngtr$	is less than or equal to; is not greater than
$>$	is greater than
$\geqslant$; $\nless$	is greater than or equal to; is not less than
∞	infinity

3. *Operations*

$a + b$	a plus b
$a - b$	a minus b
$a \times b$, ab, $a.b$	a multiplied by b
$a \div b$, $\frac{a}{b}$, a/b	a divided by b
$a : b$	the ratio of a to b
$\sum_{i=1}^{n} a_i$	$a_1 + a_2 + ... + a_n$
$\sqrt{a}$	the positive square root of the real number a
$\lvert a \rvert$	the modulus of the real number a
$n!$	n factorial for $n \in \mathbb{Z}^+ \cup \{0\}$, $(0! = 1)$
$\binom{n}{r}$	the binomial coefficient $\frac{n!}{r!(n-r)!}$, for $n, r \in \mathbb{Z}^+ \cup \{0\}$, $0 \leqslant r \leqslant n$
	$\frac{n(n-1)...(n-r+1)}{r!}$, for $n \in \mathbb{Q}$, $r \in \mathbb{Z}^+ \cup \{0\}$

4. *Functions*

f	the function f
$\mathrm{f}(x)$	the value of the function f at x
$\mathrm{f}: A \to B$	f is a function under which each element of set A has an image in set B
$\mathrm{f}: x \mapsto y$	the function f maps the element x to the element y
f^{-1}	the inverse of the function f
$\mathrm{g} \circ \mathrm{f}$, gf	the composite function of f and g which is defined by $(\mathrm{g} \circ \mathrm{f})(x)$ or $\mathrm{gf}(x) = \mathrm{g}(\mathrm{f}(x))$
$\lim\limits_{x \to a} \mathrm{f}(x)$	the limit of $\mathrm{f}(x)$ as x tends to a
Δx ; δx	an increment of x
$\dfrac{\mathrm{d}y}{\mathrm{d}x}$	the derivative of y with respect to x
$\dfrac{\mathrm{d}^n y}{\mathrm{d}x^n}$	the nth derivative of y with respect to x
$\mathrm{f}'(x), \mathrm{f}''(x), \ldots, \mathrm{f}^{(n)}(x)$	the first, second, … nth derivatives of $\mathrm{f}(x)$ with respect to x
$\int y \, \mathrm{d}x$	indefinite integral of y with respect to x
$\int_a^b y \, \mathrm{d}x$	the definite integral of y with respect to x for values of x between a and b
$\dot{x}, \ddot{x}, \ldots$	the first, second, …derivatives of x with respect to time

5. *Exponential and Logarithmic Functions*

e	base of natural logarithms
e^x, $\exp x$	exponential function of x
$\log_a x$	logarithm to the base a of x
$\ln x$	natural logarithm of x
$\lg x$	logarithm of x to base 10

6. *Circular Functions and Relations*

sin, cos, tan, cosec, sec, cot	the circular functions
$\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, cosec^{-1}, $\sec^{-1}$, $\cot^{-1}$	the inverse circular functions

7. *Complex Numbers*

i	the square root of –1
z	a complex number, $z = x + \mathrm{i}y$ $= r(\cos\theta + \mathrm{i}\sin\theta),\ r \in \mathbb{R}_0^+$ $= r\mathrm{e}^{\mathrm{i}\theta},\ r \in \mathbb{R}_0^+$
$\mathrm{Re}\, z$	the real part of z, $\mathrm{Re}\,(x + \mathrm{i}y) = x$
$\mathrm{Im}\, z$	the imaginary part of z, $\mathrm{Im}\,(x + \mathrm{i}y) = y$
$\lvert z\rvert$	the modulus of z, $\lvert x + \mathrm{i}y\rvert = \sqrt{x^2 + y^2}$, $\lvert r(\cos\theta + \mathrm{i}\sin\theta)\rvert = r$
$\arg z$	the argument of z, $\arg(r(\cos\theta + \mathrm{i}\sin\theta)) = \theta$, $-\pi < \theta \leqslant \pi$
z^*	the complex conjugate of z, $(x + \mathrm{i}y)^* = x - \mathrm{i}y$

8. *Matrices*

$\mathbf{M}$	a matrix $\mathbf{M}$
$\mathbf{M}^{-1}$	the inverse of the square matrix $\mathbf{M}$
$\mathbf{M}^{\mathrm{T}}$	the transpose of the matrix $\mathbf{M}$
det $\mathbf{M}$	the determinant of the square matrix $\mathbf{M}$

9. *Vectors*

$\mathbf{a}$	the vector $\mathbf{a}$
$\overrightarrow{AB}$	the vector represented in magnitude and direction by the directed line segment AB
$\hat{\mathbf{a}}$	a unit vector in the direction of the vector $\mathbf{a}$
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	unit vectors in the directions of the cartesian coordinate axes
$\lvert\mathbf{a}\rvert$	the magnitude of $\mathbf{a}$
$\lvert\overrightarrow{AB}\rvert$	the magnitude of $\overrightarrow{AB}$
$\mathbf{a.b}$	the scalar product of $\mathbf{a}$ and $\mathbf{b}$
$\mathbf{a}\times\mathbf{b}$	the vector product of $\mathbf{a}$ and $\mathbf{b}$

10. *Probability and Statistics*

A, B, C, etc.	events
$A \cup B$	union of events A and B
$A \cap B$	intersection of the events A and B
$\mathrm{P}(A)$	probability of the event A
A'	complement of the event A, the event 'not A'
$\mathrm{P}(A \mid B)$	probability of the event A given the event B
X, Y, R, etc.	random variables
x, y, r, etc.	value of the random variables X, Y, R, etc.
$x_1, x_2, \ldots$	observations
$f_1, f_2, \ldots$	frequencies with which the observations, $x_1, x_2, \ldots$occur
$\mathrm{p}(x)$	the value of the probability function $\mathrm{P}(X = x)$ of the discrete random variable X
$p_1, p_2, \ldots$	probabilities of the values $x_1, x_2, \ldots$of the discrete random variable X
$\mathrm{f}(x), \mathrm{g}(x)\ldots$	the value of the probability density function of the continuous random variable X
$\mathrm{F}(x), \mathrm{G}(x)\ldots$	the value of the (cumulative) distribution function $\mathrm{P}(X \leqslant x)$ of the random variable X
$\mathrm{E}(X)$	expectation of the random variable X
$\mathrm{E}[\mathrm{g}(X)]$	expectation of $\mathrm{g}(X)$
$\mathrm{Var}(X)$	variance of the random variable X
$\mathrm{B}(n, p)$	binomial distribution, parameters n and p
$\mathrm{Po}(\mu)$	Poisson distribution, mean μ
$\mathrm{N}(\mu, \sigma^2)$	normal distribution, mean μ and variance σ^2
μ	population mean
σ^2	population variance
σ	population standard deviation
$\bar{x}$	sample mean
s^2	unbiased estimate of population variance from a sample, $s^2 = \frac{1}{n-1}\Sigma\left(x-\bar{x}\right)^2$
φ	probability density function of the standardised normal variable with distribution $\mathrm{N}(0, 1)$
Φ	corresponding cumulative distribution function
ρ	linear product-moment correlation coefficient for a population
r	linear product-moment correlation coefficient for a sample